Evaluation of the plastic zone at the crack tip of strain hardening low density polyethylene

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Prediction of the plastic zone in front of the crack tip has been done using a modified Dugdale's model for strain hardening low density polyethylene. By knowing the strain hardening index n, yield stress σ_Y, crack length a, and average stress acting far away from the crack σ_∞, the size of the plastic zone, r_P, can be calculated. Simulation seems to be promising for the ratio of r_P to the ligament length, b, falling in the range of 6–24%. If r_P/b is $<6\%$, restriction from the elastic domain seems to affect the expansion of the plastic zone. For r_P/b $>24\%$, the plastic zone becomes so large that the HRR stress singularity together with the boundary condition used in this simulation may lose its significance and the resultant stress distribution deviates from the real situation. A detailed evaluation of the stress function inside the plastic zone is needed before further comment can be made.

(Keywords: strain hardening; crack; plastic zone; low density polyethylene; stress singularity)

INTRODUCTION

One of the main subjects in fracture mechanics has involved characterizing the domain of the plastic zone in front of the crack tip. According to the work of Broberg[1] and Andersson[2], the plastic zone plays an important role in stable crack growth. Combination of the von Mises' criterion of yielding and the stress function derived from linear elastic fracture mechanics (LEFM) is valid only for materials showing linear mechanical behaviour and undergoing very small scale yielding. While most materials possess, to some extent, non-linear mechanical behaviour, the plastic zone calculated from LEFM is too small and not realistic. This is especially true for polymeric materials. For polymers it is common for yield stress to be much lower than the maximum stress (stress at break). It is reasonable to say that, inside the plastic zone, material does not fracture immediately after yielding. That is to say, at the crack tip there exists a yielded domain which is not necessarily small compared with other dimensions. Small scale yielding at the crack tip is no longer convincing.

Effort has been focused on modelling the crack tip stress field assuming either perfectly elastic or perfectly plastic mechanical behaviour. Irwin[3] suggested that the stress distribution in front of the crack tip can be expressed by assuming that the crack length is longer than that in the real situation. With his method, the plastic zone calculated was found to be twice that calculated from LEFM under plane stress conditions. Dugdale[4] proposed his well known model by assuming that the crack tip front can be treated as two isolated singular domains acted on by a remote stress and by a compressive yield stress, respectively. Dugdale's model has been proved to be quite successful for materials with perfectly plastic behaviour after yielding. Both methods faced a common difficulty in that the stress singularity in the plastic zone is not yet specifically derived. As perfectly plastic behaviour after yielding is not a common phenomenon in engineering plastics, Dugdale's model has to be modified. While Irwin's method took care of the stress distribution in the crack tip front, this distribution is based on the assumption of linear mechanical behaviour as used in LEFM so that material undergoing non-linear mechanical behaviour fails using this method.

For materials with elastic–plastic behaviour, the stress, or strain, singularity around the crack tip has been extensively studied. Unfortunately, evaluation of the stress distribution is not clear except for anti-plane shearing[5]. Direct application of either von Mises' or Tresca's criterion of yielding is impossible without knowing the stress components. We have studied this problem and proposed a modified model for materials showing strain hardening behaviour after yielding. The main point is that strain hardening has been taken into account so that the stress inside the plastic zone is no longer constant as in perfectly plastic materials. This method has been applied to low density polyethylene (LDPE) and the result is encouraging.

THEORY

Hutchinson[6,7] and Rice and Rosengren[8] suggested that for power-law hardening material, the singular behaviour

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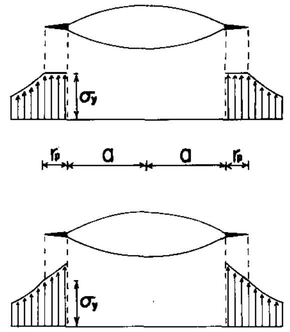

Figure 1 Stress distribution of the Dugdale model (upper) and of the modified model (lower)

at the crack tip could be expressed by the following equation:

$$\sigma_{ij} = k_{ij}\sigma_{ij}(\theta, n) r^{(-1/n+1)} \quad (1)$$

where r is the distance from the crack tip and n is the strain hardening index. The term k_{ij}, which is the stress intensity function and is a scalar in LEFM, is now a tensor. Hence $\sigma_{ij}(\theta, n)$ is a function of θ, the angle from the crack plane, and n. This is known as the HRR stress singularity. While equation (1) involves many variables, it has to be simplified for further application. First, we focus on the restriction that $\theta = 0$. That is to say, we will only deal with this problem in the crack plane with its normal parallel to the stretch direction. Let us define the stretch direction as the y-axis. Second, as the boundary condition used by Irwin, we assume that σ_{yy}, which is the stress component in the stretch direction, equals the yield stress σ_Y at the boundary between the plastic zone and the remaining elastic field. If this boundary is at the position r_P, that is, the size of the plastic zone at the crack tip front equals r_P, equation (1) can be rewritten as

$$\sigma_Y = k_{yy}\sigma_{yy}(0, n) r_P^{(-1/n+1)} \quad (1')$$

so that

$$k_{yy}\sigma_{yy}(0, n) = \sigma_Y r_P^{(1/n+1)} \quad (2)$$

By combining equations (1′) and (2), we obtain

$$\sigma_{yy} = \sigma_Y r_P^{(1/n+1)} r^{(-1/n+1)} \quad (3)$$

for $\theta = 0$.

In equation (3), it appears that the stress distribution is autonomous inside the plastic zone. In fact, as will be shown later, r_P in equation (3) is determined by the crack length a and the remote stress σ_∞. This, in turn, affects the stress distribution inside the plastic zone.

As shown in *Figure 1*, Dudgale's model is modified by replacing the constant yield stress by a stress field distributing according to the singularity given in equation (3). The stress intensity factor at the crack tip due to the compressive stress can be derived using Irwin's stress function. For a centre cracked specimen with indefinite width as shown in *Figure 2*, Irwin's stress function[9] Z_I can be expressed as:

$$dZ_I = \frac{p(\xi)\,d\xi}{\Pi(z-\xi)}\left(\frac{\sqrt{a^2-\xi^2}}{\sqrt{z^2-a^2}}\right)$$

The stress intensity factor at the crack tip due to this stress function is:

$$\begin{aligned}(dK_I)_a &= \sqrt{2\Pi}\lim_{z\to a}[(\sqrt{z-a})dZ_I] \\ &= \sqrt{2\Pi}\lim_{z\to a}\left\{(\sqrt{z-a})\left[\frac{p(\xi)d\xi}{\Pi(z-\xi)}\right] \times \left(\frac{\sqrt{a^2-\xi^2}}{\sqrt{z^2-a^2}}\right)\right\} \\ &= \frac{p(\xi)}{\sqrt{\Pi a}}\left(\frac{\sqrt{a+\xi}}{\sqrt{a-\xi}}\right)d\xi \end{aligned} \quad (4)$$

Therefore, the stress intensity factor at the crack tip due to the distributing stress in the positive domain is:

$$(K_I)_{a+} = \int_b^a \frac{p(\xi)}{\sqrt{\Pi a}}\left(\frac{\sqrt{a+\xi}}{\sqrt{a-\xi}}\right)d\xi \quad (5)$$

and that due to the distributing stress in the negative domain can be derived by the same procedure and can be expressed as:

$$(k_I)_{a-} = \int_{-a}^{-b} \frac{p(\xi)}{\sqrt{\Pi a}}\left(\frac{\sqrt{a+\xi}}{\sqrt{a-\xi}}\right)d\xi \quad (6)$$

Then the total intensity factor at the crack tip in the positive domain is:

$$\begin{aligned}(K_I)_a &= (K_I)_{a+} + (K_I)_{a-} \\ &= \int_b^a \frac{p(\xi)}{\sqrt{\Pi a}}\left(\frac{\sqrt{a+\xi}}{\sqrt{a-\xi}}\right)d\xi \\ &\quad + \int_{-a}^{-b} \frac{p(\xi)}{\sqrt{\Pi a}}\left(\frac{\sqrt{a+\xi}}{\sqrt{a-\xi}}\right)d\xi \end{aligned} \quad (7)$$

On the other hand, the stress intensity factor due to the remote stress can be expressed as:

$$(K_I)_\infty = \sigma_\infty\sqrt{\Pi a} \quad (8)$$

When Dugdale's model is applied, the crack length a in equations (7) and (8) should be changed to $a + r_P$ where r_P is the size of the plastic zone. As the stress at the pseudo-crack tip ($a + r_P$) is constant (σ_Y), the summation of the stress factors from σ_∞ and from distributing stress at the tip should be zero. This leads to the following equation:

$$(K_I)_\infty = (K_I)_a = (K_I)_{a+} + (K_I)_{a-} \quad (9)$$

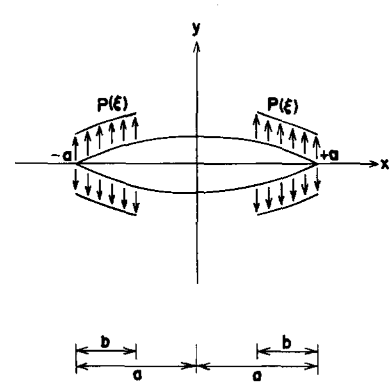

Figure 2 Distributed stress around the crack tips

By replacing $p(\xi)$ in equation (9) with σ_{yy} in equation (3), the plastic zone (r_P) can be derived from equation (9) by simple numerical analysis. As can be seen, if $p(\xi)$ is replaced by the constant distributing stress σ_Y, equation (9) can be reduced to the normal Dugdale model.

In the case of single edge-cracked tension (SECT), the above model can be modified by assuming that the crack length is very large so that the effect from the negative domain can be neglected. Because the plastic zone is small compared with the crack length, the value of (ξ/a) should be close to unity. Then equation (4) becomes

$$(\mathrm{d}K_I)_a = \frac{p(\xi)}{\sqrt{\Pi a}}\left(\frac{\sqrt{a+\xi}}{\sqrt{a-\xi}}\right)\mathrm{d}\xi = \frac{p(\xi)}{\sqrt{\Pi a}}\left[\frac{1+\xi/a}{\sqrt{1-(\xi/a)^2}}\right]\mathrm{d}\xi \doteqdot \frac{p(\xi)}{\sqrt{\Pi a}}\left[\frac{2}{\sqrt{1-(\xi/a)^2}}\right]\mathrm{d}\xi \qquad (10)$$

In the case of a SECT specimen with definite dimensions, equation (10) can be modified further by multiplying the right-hand terms with a factor $f(\xi/a)$, which varies[9] from 1.3 to 1 depending on the position ξ. If the plastic zone is not large compared with the crack length, this factor is in the range of 1.1–1. Therefore the stress intensity factor at the crack tip of a SECT specimen can be expressed as:

$$(K_I)_a = \int_b^a f(\xi/a)\left[\frac{p(\xi)}{\sqrt{\Pi a}}\right]\left[\frac{2}{\sqrt{1-(\xi/a)^2}}\right]\mathrm{d}\xi \qquad (11)$$

while the stress intensity factor due to σ_∞ is still that shown in equation (8). By applying Dugdale's model, from equations (3), (8) and (11), we obtain:

$$\sigma_\infty\sqrt{\Pi(a+r_P)} - \int_a^{a+r_P} \frac{\sigma_Y r_P^{(1/n+1)}}{\sqrt{\Pi(a+r_P)}} \left\{\frac{2(\xi-a)^{(-1/n+1)}}{\sqrt{1-[\xi/(a+r_P)]^2}}\right\} f(\xi/a+r_P)\,\mathrm{d}\xi = 0 \qquad (12)$$

Again, r_P in equation (12) can be derived by a numerical method.

EXPERIMENTAL

The LDPE used in this work was supplied by the Mitsubishi Petro. Chem. Co. (trade no. PK-30). As shown in *Figure 3* this material shows strain hardening after yielding and no necking behaviour is observed. *Figure 4* is plotted by normalizing the stress and strain

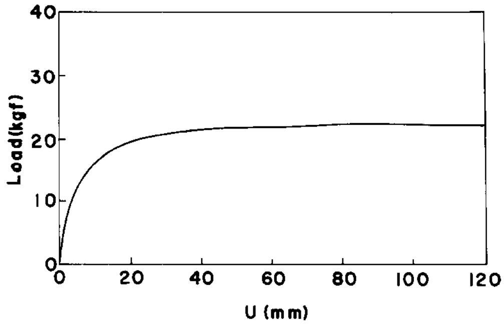

Figure 3 Load–displacement curve of LDPE

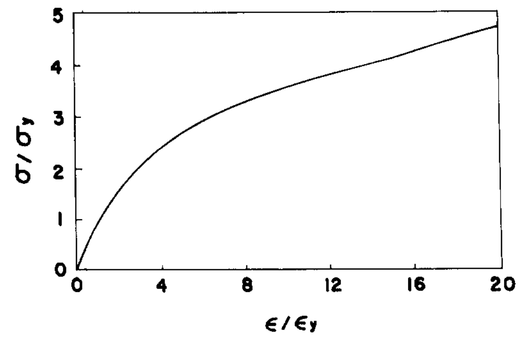

Figure 4 σ/σ_Y *versus* $\varepsilon/\varepsilon_Y$ of LDPE

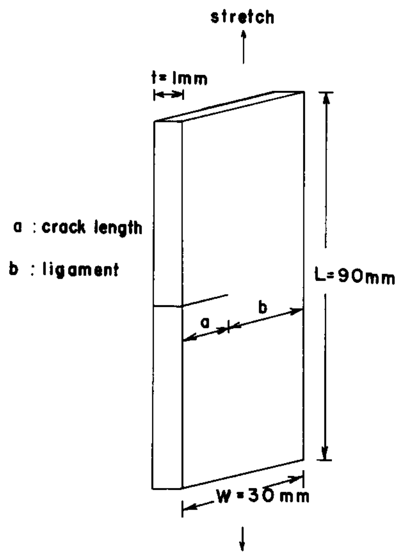

Figure 5 Specimen dimensions of LDPE

to non-dimensional terms by the yield stress and the strain at the yield point. The yield stress is found to be 5.68 MPa. The strain hardening index n from the normalized curve is found to be 1.54 with a deviation of 0.25 due to the following stress–strain relation:

$$\frac{\sigma}{\sigma_{yy}} = \left(\frac{\varepsilon}{\varepsilon_{yy}}\right)^{-n} \qquad (13)$$

SECT samples were prepared according to the dimensions shown in *Figure 5*. According to our previous work[10], the length of the sample has to be three times the value of the width in order to prevent boundary effects from the grips. The crack length varies from 9 to 21 mm so that the ratio of the initial crack length a to the width w is in the range of 0.3–0.7. The test was performed at a tensile speed of 0.4 mm min^{-1} at room temperature.

Photographs were taken every millimetre of stretching by using a polarizer and an analyser so that the isochromatic fringe loops of the samples which implied the iso-stress path could be characterized. The iso-stress path of plane stress is circular while that of plane strain is shaped like two leaves. This can be easily proved by replacing the yield stress in von Mises' yielding criterion with any particular stress. In *Figure 6a* the isochromatic fringe loops close to the crack tip are circular while the remote ones are of the plane strain type. It is found that the thickness in the plane stress area decreased a lot which was not recoverable even after unloading. In this work, the intersect between the plane stress fringes and

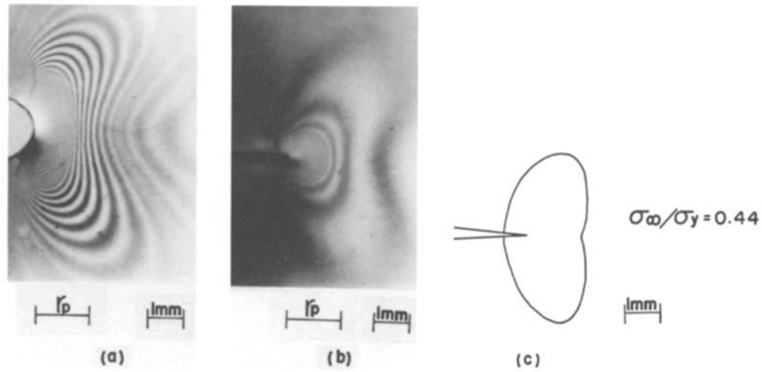

Figure 6 Schematic representation of the plastic zone with initial crack length of 12 mm: (a) during loading; (b) after unloading; (c) simulation from FEM calculated at a stress level similar to that of (a) and (b)

the plane strain fringes is defined as the boundary of the plastic zone. This has been confirmed by unloading the samples to zero load and the plastic zone showing the plane stress isochromatic loop was found to be unrecoverable (*Figure 6b*). The shape of the plastic zone is similar to that calculated using the Finite Element Method (FEM) by Andersson[2] on poly(vinyl chloride) under plane stress conditions. Our FEM result is shown in *Figure 6c* for comparison. As the crack propagated with the displacement control deformation, it was found that the plastic zone also expanded. Evaluation of the plastic zone is not limited to stationary cracks but also extends to growing cracks.

RESULTS AND DISCUSSION

As shown in equation (3), if $n = 1$, all the discussion converges to the perfectly elastic type. On the other hand, if $n = \infty$, then we are dealing with materials with perfectly plastic behaviour after yielding. The former condition can be dealt with by LEFM under plane stress conditions while the latter corresponds to Dugdale's model. Obviously, materials showing strain hardening should fall in between. *Figure 7* shows how the non-dimensional ratio of the plastic zone to the crack length, r_P/a, changes with the non-dimensional ratio of the remote stress to the yield stress, σ_∞/σ_Y, in the elastic and plastic cases. The solid lines are calculated from the modified model according to equation (13). It is found that for $\sigma_\infty/\sigma_Y < 0.3$, the curve with $n = 1$ fits very well to values calculated by LEFM under plane stress conditions according to the following equation:

$$\frac{r_P}{a} = \frac{1}{2}\left(\frac{\sigma_\infty}{\sigma_Y}\right)^2 \tag{14}$$

The perfectly plastic case ($n = 10^6$) reveals that the relationship between r_P/a and σ_∞/σ_Y is similar to Dugdale's model according to the following equation:

$$\frac{r_P}{a} = \frac{\Pi^2}{8}\left(\frac{\sigma_\infty}{\sigma_Y}\right)^2 \tag{15}$$

The experimental results are shown in *Figure 8* with non-dimensional coordinates σ_∞/σ_Y and r_P/a. Three curves calculated with $n = 1$, 1.54 and 10^6 are presented. As seen from *Figure 8*, the experimental results are in agreement with the calculated curve with $n = 1.54$. The prediction based on perfectly plastic behaviour ($n = 10^6$) shows quite a large deviation from the real situation. As

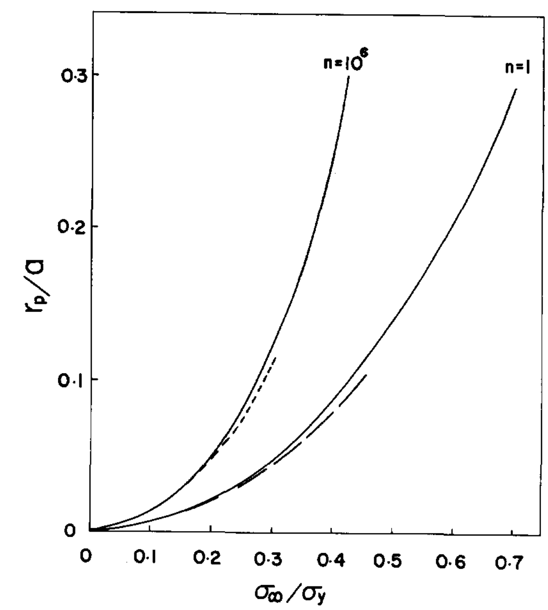

Figure 7 Comparison of r_P/a calculated from LEFM (——), the Dugdale model (---) and from the modified model (—)

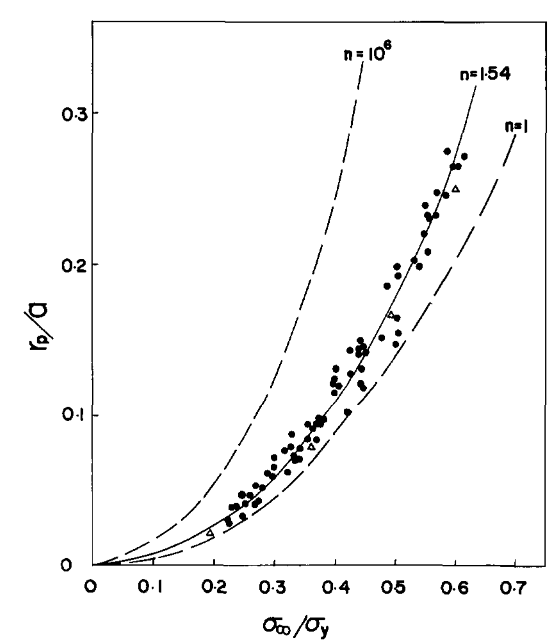

Figure 8 Comparison of experimental results with those calculated for the modified model

LDPE shows strain hardening after yielding, the stress inside the plastic zone should be much higher than the yield stress. The maximum stress of LDPE is ~ 19.5 MPa which is close to four-fold σ_Y. This fact reveals that after the stress of any point of the sample reaches σ_Y, it will continue to increase instead of remaining constant. As a consequence, for the whole system, more energy is consumed for the deformation inside the plastic zone so that the plastic zone expands slower than prediction by assuming perfectly plastic behaviour. The triangular symbols are results from FEM calculation for a crack length of 12 mm. It can be seen that the numerical calculation fits quite well with the modified model.

As shown in *Figure 8*, some of the points fall on the curve by assuming perfectly elastic behaviour. The most interesting thing is that these points are from samples with $a/w < 0.4$. These samples possess a larger ligament length b compared with other samples. We find that if r_P/b is $<6\%$, the experimental results are close to the curve with $n = 1$. This can be explained by considering that the elastic field surrounding the plastic zone will restrict the development of the plastic zone if the r_P/b ratio is quite small. The stress outside the plastic zone is less than the yield stress σ_Y. In order to increase the stress level of the elastic part, the stress in the plastic zone must reach a higher level first. According to LEFM, for the same σ_∞, samples with smaller crack lengths result in smaller stress intensity functions and the stress distribution in front of the crack tip should be lower. Since a/w is <0.4, the crack length is small so that the stress intensity at the crack tip is also small. As a consequence, the yielded domain at the crack tip is small. This is especially true when the ligament is much larger than the yielded domain. In this case, restriction from the elastic field surrounding the plastic zone is strong. As r_P/b becomes $> \sim 6\%$, the experimental data correspond to the curve with $n = 1.54$. This implies that the restriction from the elastic part is smaller and the size of the plastic zone becomes predictable.

The calculation is based on the assumption that the stress inside the plastic zone is dominated by the HRR stress singularity. As the plastic zone becomes very large, this assumption seems to be no longer valid. Indeed, this has been confirmed by carefully tracing the growth of the plastic zone. We find that if r_P/b becomes $> \sim 24\%$, some of the experimental results deviate from the predicted curve. This happens before the increase of the load becomes slow in the load–displacement curve and the propagation of the crack is no longer small. We suggest that this is due to the fact that the stress in the elastic domain reaches a certain high level and the plastic zone is no longer small compared with the elastic part. Even the increment in σ_∞ becomes slower, and the increasing crack length which, in turn, results in larger stress intensity function becomes another potential factor in increasing the stress level. For the material out of the plastic zone, the transition from the elastic into the plastic state becomes faster. Also the stress distribution inside the plastic zone is no longer controlled by the HRR singularity[6–8]. Another important effect that has to be taken into account is the existence of a large plastic zone that is not a 'thin layer' as Dugdale's crack. Then the stress intensity function $(K_I)_\infty$ should be different from that in equation (8). Because the exact stress distribution in the plastic zone is not available, we are unable to further predict the growth of the plastic zone. Detailed evaluations of the stress singularity of both the large plastic zone and the elastic field surrounding it have to be studied before further comment can be made. It is well known that at the crack tip a fracture processing zone exists and stress relaxation occurs. This 'end-zone'[1] is very small compared with the other dimensions. It is found that if 5% of r_P is taken as the fracture processing zone and the stress is kept constant inside it, the calculation does not change significantly.

CONCLUSIONS

Prediction of the plastic zone at the crack tip that is not limited to small scale yielding has been performed on strain hardening LDPE. Our calculation is based on the Dugdale model with HRR stress singularity. Four variables (σ_∞, σ_Y, n and a) are needed for the calculation while results can be represented by two non-dimensional terms (r_P/a and σ_∞/σ_Y). Simulation is found to be good not only for stationary cracks but also for propagating cracks. For shallow cracks with $r_P/b < 6\%$, experimental results are close to calculated values by assuming perfectly elastic behaviour. This is interpreted by considering the restriction from the elastic part surrounding the plastic zone. In this case, 'small scale yielding' seems to be the dominate mechanism. While for r_P/b $>24\%$, the observed plastic zone exceeds the calculated one. In this case, the stress distribution inside the plastic zone may not be as for HRR singularity. Also, as the plastic zone is not small compared with the ligament length, the plastic zone may affect the stress intensity function calculated by the pseudo-crack length ($a + r_P$).

REFERENCES

1 Broberg, K. B. *Int. J. Frac. Mech.* 1968, **4**, 11
2 Andersson, H. *J. Mech. Phys. Solids* 1973, **21**, 337
3 Irwin, G. R. 'Fracture, Handbuch der Physik VI', (Ed. Flugge), Springer, Berlin, 1956, p. 551
4 Dugdale, D. S. *J. Mech. Phys. Solids* 1960, **8**, 100
5 Saeedvafa, M. and Rice, J. R. *J. Mech. Phys. Solids* 1989, **37**, 673
6 Hutchinson, J. W. *J. Mech. Phys. Solids* 1968, **16**, 13
7 Hutchinson, J. W. *J. Mech. Phys. Solids* 1968, **16**, 337
8 Rice, J. R. and Rosengren, G. F. *J. Mech. Phys. Solids* 1968, **16**, 1
9 Tada, H., Paris, P. C. and Irwin, G. R. 'The Stress Analysis of Cracks Handbook', Del Research Corporation, 1973
10 Amano, M., Wang, M., Hibi, S., Mori, T. and Kadata, M. *Kobunshi Ronbunshu* 1990, **47**, 537